Topic 10 -
More on Linear Transformations

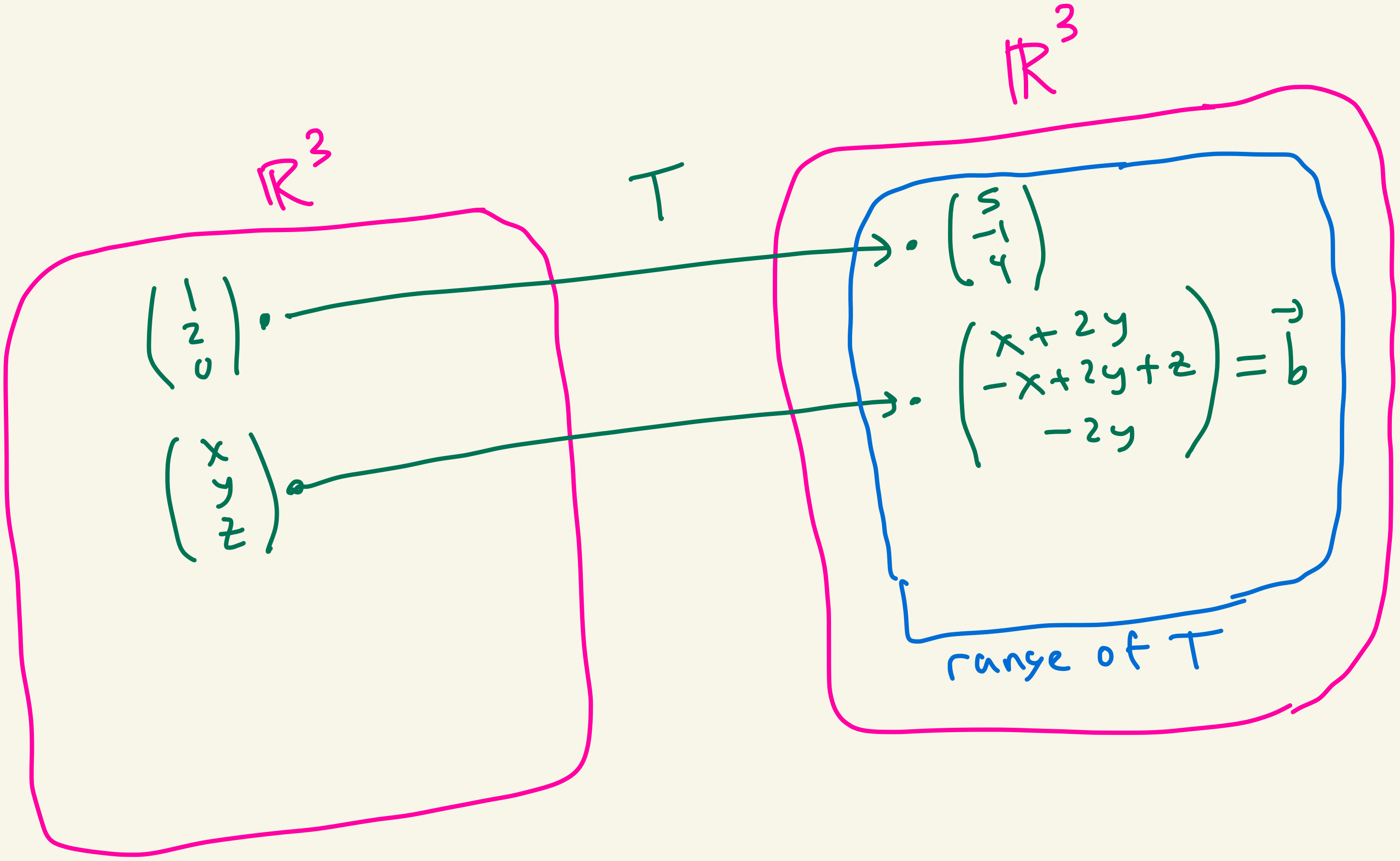
Suppose you the a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & -2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+2y \\ -x+y+z \\ -2y \end{pmatrix}$$

You can ask the question: What is the range of this function? Ie, what vectors $\vec{b}$ in $\mathbb{R}^3$ can we get when we plug $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ into T?

For example, $T\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+4 \\ -1+2+0 \\ -4 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix}$

So, $\begin{pmatrix} 5 \\ 1 \\ -4 \end{pmatrix}$ is in the range of T.



Note that

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & -2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} x + 2y \\ -x + y + z \\ -2y \end{pmatrix}$$

$$= \begin{pmatrix} x \\ -x \\ 0 \end{pmatrix} + \begin{pmatrix} 2y \\ y \\ -2y \end{pmatrix} + \begin{pmatrix} 0 \\ z \\ 0 \end{pmatrix}$$

$$= x\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} + z\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Thus, the vectors $\vec{b}$ in the range of T consist of all linear combinations of the columns of T. That is, the range of T is the space spanned by the columns of T.

It turns out this space is related to the "nullspace" of T which is all the vectors $\vec{v}$ where $T(\vec{v}) = \vec{0}$.

For simplicity, since $T(\vec{v}) = A\vec{v}$ we will just discuss A by itself below and the discussion necessarily applies to T.

<u>Def:</u> Let A be matrix.
The solutions $\vec{x}$ to the equation
$A\vec{x} = \vec{0}$ form the <u>nullspace</u> of A.
The space spanned by the columns
of A is called the <u>column space</u>
of A. We denote the nullspace
of A by $N(A)$. We denote
the column space of A by $R(A)$

<u>Theorem:</u> If A is $m \times n$ then
$N(A)$ is a subspace of $\mathbb{R}^n$ and
$R(A)$ is a subspace of $\mathbb{R}^m$.

Def: The nullity of A is defined to be the dimension of the nullspace of A.

The rank of A is defined to be the dimension of the column space of A.

Ex: Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix}$.

Let's find some vectors in the nullspace of A.

$$\underbrace{\overbrace{\begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix}}^{A}}_{2\times 3} \underbrace{\overbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}^{\vec{x}}}_{3\times 1} = \overbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}^{\vec{0}}$$

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We need to find $\vec{x}$'s that solve the above $A\vec{x} = \vec{0}$.

If $\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then

$$A\vec{x} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (1)(0)+(0)(0)+(-1)(0) \\ (2)(0)+(0)(0)+(-2)(0) \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So, $\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is in the nullspace of A.

If $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, then

$$A\vec{x} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} (1)(1) + (0)(1) + (-1)(1) \\ (2)(1) + (0)(1) + (-2)(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So, $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is in the nullspace of A.

Let's find some vectors in the column space of A.

Recall that the column space is the subspace spanned by the columns of A.

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix}$$

columns of A are: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

A vector in the column space of A has the form

$$a\begin{pmatrix} 1 \\ 2 \end{pmatrix} + b\begin{pmatrix} 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

where a, b, c can be any real numbers.
For example if $a = 5$, $b = 25$, $c = 12$ then we get

$$\begin{pmatrix} -7 \\ -14 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 25\begin{pmatrix} 0 \\ 0 \end{pmatrix} + 12\begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

So, $\begin{pmatrix} -7 \\ -14 \end{pmatrix}$ is in the column space of A.

If $a=1$, $b=10^6$, $c=2$, then we get

$$\begin{pmatrix}-1\\-2\end{pmatrix} = 1\cdot\begin{pmatrix}1\\2\end{pmatrix} + 10^6\cdot\begin{pmatrix}0\\0\end{pmatrix} + 2\cdot\begin{pmatrix}-1\\-2\end{pmatrix}$$ $\boxed{\text{pg } 6}$

So, $\begin{pmatrix}-1\\-2\end{pmatrix}$ is in the column space of A.

Again, recall from our previous discussion that

A vector in the column space has the form:

$$a\begin{pmatrix}1\\2\end{pmatrix} + b\begin{pmatrix}0\\0\end{pmatrix} + c\begin{pmatrix}-1\\-2\end{pmatrix}$$

$$= \begin{pmatrix}1\cdot a\\2\cdot a\end{pmatrix} + \begin{pmatrix}0\cdot b\\0\cdot b\end{pmatrix} + \begin{pmatrix}-c\\-2c\end{pmatrix}$$

$$= \begin{pmatrix}1\cdot a + 0\cdot b + (-1)c\\2\cdot a + 0\cdot b + (-2)c\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0 & -1\\2 & 0 & -2\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = A\begin{pmatrix}a\\b\\c\end{pmatrix}$$

So, $\vec{d}$ is in the column space of A if there exists a vector $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ where $A\vec{x} = \vec{d}$.

For example, from above we got

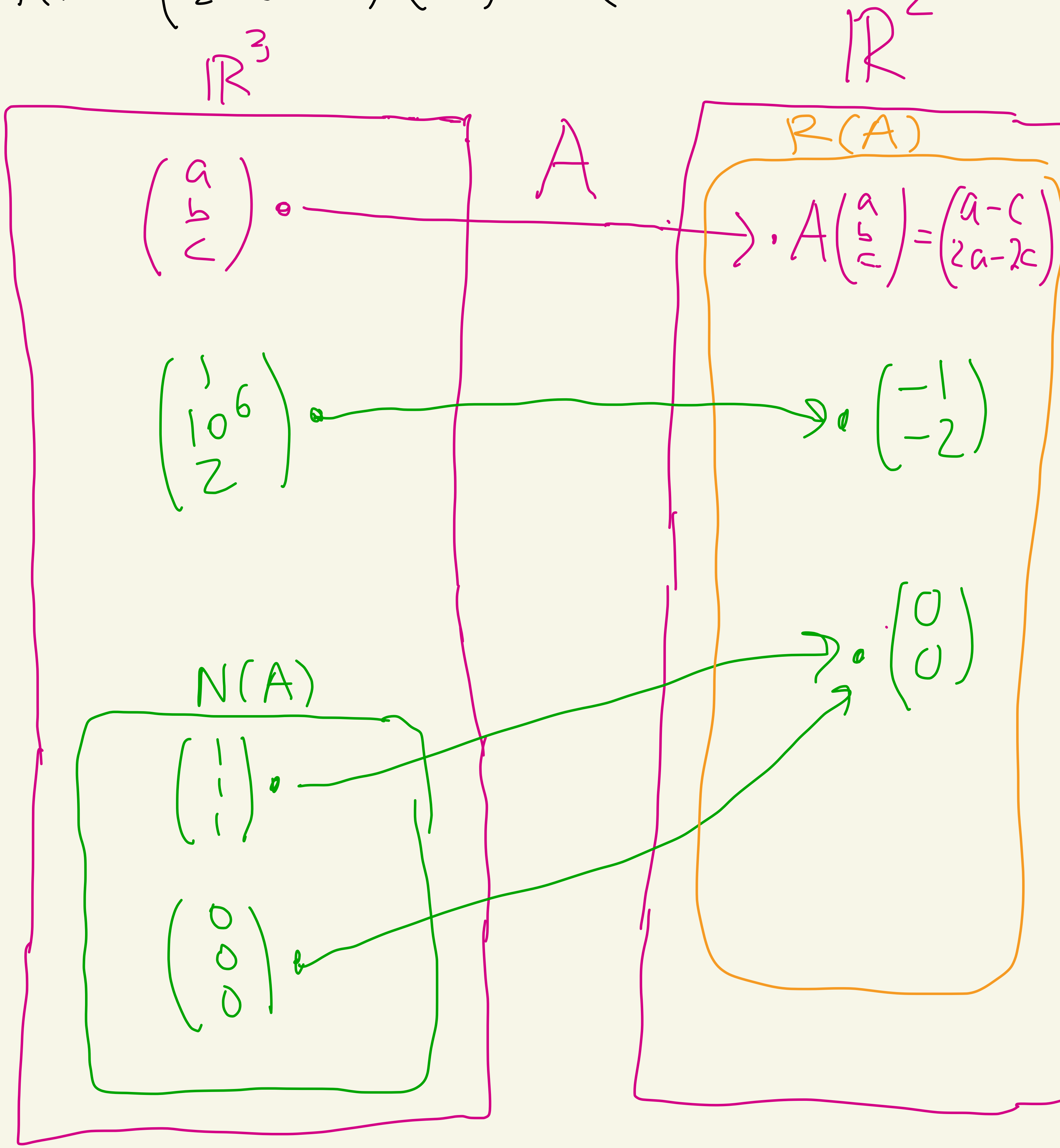
$$\underbrace{\begin{pmatrix} -1 \\ -2 \end{pmatrix}}_{\vec{b}} = 1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 10^6 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 \\ 10^6 \\ 2 \end{pmatrix}}_{\vec{x}}$$

As before, we can think of A as a linear transformation that takes vectors $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ from $\mathbb{R}^3$ and outputs vectors $A\vec{x}$ in $\mathbb{R}^2$ the the column space of A is the range of this function.

Here's a picture.

$$A\vec{x} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a-c \\ 2a-2c \end{pmatrix}.$$



Here is a theorem to help us find a basis for the column space

<u>Theorem:</u> Let A be a matrix. Reduce A down to row-echelon form, suppose R is that reduced matrix.

The columns of A that correspond to the columns of R that contain the leading 1's in R form a basis for the column space of A.

__Ex:__ Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix}$

Find bases for $N(A)$ and $R(A)$.
Find the nullity and rank of A.

$\boxed{\text{Let's find the column space } R(A)}$

$$\underbrace{\begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix}}_{A} \xrightarrow{-2R_1+R_2 \to R_2} \underbrace{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}}_{R}$$

$R = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ ← circle columns in R w/ leading 1's

$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix}$ ← circle the corresponding columns of A

Thus,

def of R(A)

$$R(A) = \text{span}\left(\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}-1\\-2\end{pmatrix}\right\}\right)$$

$$= \text{span}\left(\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}\right)$$

what we just calculated

Basis for $R(A)$ is $\begin{pmatrix}1\\2\end{pmatrix}$.

Why did this happen?

If $\vec{v}$ is in $R(A)$ above then

$$\vec{v} = c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}0\\0\end{pmatrix} + c_3\begin{pmatrix}-1\\-2\end{pmatrix}$$

$$= \begin{pmatrix}c_1 - c_3\\2c_1 - 2c_3\end{pmatrix}$$

$$= (c_1 - c_3)\begin{pmatrix}1\\2\end{pmatrix}$$

Since a basis for $R(A)$ has one vector in it, the rank of A is $\dim(R(A)) = 1$.

Now let's work on $N(A)$.
We need to find all vectors $\vec{x}$ where $A\vec{x} = \vec{0}$.

$$\underbrace{\overbrace{\begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix}}^{A}}_{2\times 3} \underbrace{\overbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}^{\vec{x}}}_{3\times 1} = \underbrace{\overbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}^{\vec{0}}}_{2\times 1}$$

This becomes

$$\begin{pmatrix} x - z \\ 2x - 2z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This gives

$$\boxed{\begin{aligned} x \quad - z &= 0 \\ 2x \quad - 2z &= 0 \end{aligned}}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 2 & 0 & -2 & 0 \end{array}\right) \xrightarrow{-2R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This gives

$$\boxed{\begin{aligned} x \quad - z &= 0 \\ 0 &= 0 \end{aligned}} \quad (1)$$

leading variables

x

free variables

y, z

$$\boxed{\begin{aligned} x &= z \quad (1) \\ y &= s \quad (2) \\ z &= t \quad (3) \end{aligned}} \Rightarrow \boxed{\begin{aligned} x = z &= t \\ y &= s \\ z &= t \end{aligned}}$$

pg 15

So,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ s \\ t \end{pmatrix} = \begin{pmatrix} t \\ 0 \\ t \end{pmatrix} + \begin{pmatrix} 0 \\ s \\ 0 \end{pmatrix}$$

$$= t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

So, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ span $N(A)$.

You can verify that $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are linearly independent.

Thus, a basis for $N(A)$ is $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Therefore, the nullity of A is

$$\dim(N(A)) = 2.$$

Note:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{pmatrix} \text{ is } 2 \times 3$$

$3 = \#$ of columns

$$3 = 2 + 1$$

$$\begin{pmatrix} \# \text{ of} \\ \text{columns} \\ \text{of } A \end{pmatrix} = \boxed{\text{nullity of } A} + \boxed{\text{rank of } A}$$

pg 17

Ex: Same question but for

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{pmatrix}$$

Let's do N(A) first

$$\underbrace{\begin{pmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{pmatrix}}_{3 \times 3} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{3 \times 1} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{3 \times 1} \leftarrow \boxed{A\vec{x} = \vec{0}}$$

This becomes

$$\begin{pmatrix} x - y + 3z \\ 5x - 4y - 4z \\ 7x - 6y + 2z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

That is

$$\boxed{\begin{aligned} x - y + 3z &= 0 \\ 5x - 4y - 4z &= 0 \\ 7x - 6y + 2z &= 0 \end{aligned}}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 5 & -4 & -4 & 0 \\ 7 & -6 & 2 & 0 \end{array}\right) \xrightarrow[-7R_1+R_3\to R_3]{-5R_1+R_2\to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 1 & -19 & 0 \end{array}\right)$$

$$\xrightarrow{-R_2+R_3\to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -19 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This gives

$$\boxed{\begin{aligned} x - y + 3z &= 0 \quad (1) \\ y - 19z &= 0 \quad (2) \\ 0 &= 0 \end{aligned}}$$

leading variables
x, y

free variables
z

We get

$$\boxed{\begin{aligned} x &= y - 3z \\ y &= 19z \\ z &= t \end{aligned}} \quad \begin{matrix} (1) \\ (2) \\ (3) \end{matrix} \Rightarrow \boxed{\begin{aligned} &(3)\ z = t \\ &(2)\ y = 19z = 19t \\ &(1)\ x = y - 3z \\ &\qquad = 19t - 3t \\ &\qquad = 16t \end{aligned}}$$

Thus, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is in $N(T)$ if

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 16t \\ 19t \\ t \end{pmatrix} = t \begin{pmatrix} 16 \\ 19 \\ 1 \end{pmatrix}$$

So, $\begin{pmatrix} 16 \\ 19 \\ 1 \end{pmatrix}$ spans $N(T)$.

You can check this is a lin. ind. set because if $c_1 \begin{pmatrix} 16 \\ 19 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

then $\begin{pmatrix} 16c_1 \\ 19c_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $c_1 = 0$.

Thus a basis for $N(T)$ is $\begin{pmatrix}16\\19\\1\end{pmatrix}$

So, $\dim(N(T)) = 1$.

<u>Let's find a basis for $R(T)$</u>

We already reduced A above. Like this:

$$\underbrace{\begin{pmatrix}1 & -1 & 3\\ 5 & -4 & -4\\ 7 & -6 & 2\end{pmatrix}}_{A} \xrightarrow[-7R_1+R_3\to R_3]{-5R_1+R_2\to R_2} \begin{pmatrix}1 & -1 & 3\\ 0 & 1 & -19\\ 0 & 1 & -19\end{pmatrix}$$

$$\xrightarrow{-R_2+R_3\to R_3} \underbrace{\begin{pmatrix}1 & -1 & 3\\ 0 & 1 & -19\\ 0 & 0 & 0\end{pmatrix}}_{R}$$

So we have

$$R = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{pmatrix}$$ ← circle the columns of R with leading 1's

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{pmatrix}$$ ← circle the corresponding columns in A

A basis for the column space is $\begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \\ -6 \end{pmatrix}$.

Thus the rank of A is $\dim(R(A)) = 2$.

Note: $3 = 1 + 2$

$$\begin{pmatrix} \text{\# of} \\ \text{columns} \\ \text{of } A \end{pmatrix} = \begin{pmatrix} \text{nullity} \\ \text{of } A \end{pmatrix} + \begin{pmatrix} \text{rank} \\ \text{of } A \end{pmatrix}$$

Note:

$$\underbrace{\begin{pmatrix} 3 \\ -4 \\ 2 \end{pmatrix}}_{\text{3rd column of A}} = -16 \underbrace{\begin{pmatrix} 1 \\ 5 \\ 7 \end{pmatrix}}_{\text{1st column}} - 19 \underbrace{\begin{pmatrix} -1 \\ -4 \\ -6 \end{pmatrix}}_{\text{2nd column}}$$

this explains why we didn't need it in the basis for $R(A)$.

Theorem (Rank-Nullity Theorem)

Let A be an $m \times n$ matrix.

Then,

$$m = \dim(N(A)) + \dim(R(A))$$

$$\begin{pmatrix} \# \\ \text{columns} \end{pmatrix} = \text{nullity}(A) + \text{rank}(A)$$

(Maybe skip this in class)

Ex: Suppose that A is a matrix where a basis for its column space is

$$\left\{ \begin{pmatrix} 1 \\ 5 \\ 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Also suppose that A has 6 columns. Find the nullity of A.

Solution: We will use the rank/nullity theorem which says

$$\underbrace{6}_{\substack{\#\text{ columns} \\ \text{of } A}} = \underbrace{\text{rank}(A)}_{\substack{\text{dimension} \\ \text{of column} \\ \text{space of } A}} + \underbrace{\text{nullity}(A)}_{\substack{\text{dimension} \\ \text{of nullspace} \\ \text{of } A}}$$

From above a basis for the column space has 2 elements. So, $\text{rank}(A) = 2$. Thus $\text{nullity}(A) = 6 - \text{rank}(A) = 6 - 2 = 4$. ▨